Effects of reinjection on the scaling property of intermittency

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We study the effect of the reinjection probability distribution (RPD) on the scaling property of intermittency via the renormalization group approach. When the lower bound of reinjection is below the tangent point, the critical exponent has the well known form of $(z-1)/z$ independent of the RPD. On the other hand, when the lower bound of reinjection is at the tangent point, the critical exponent is $(z+\gamma-2)/z$ if the RPD is of an algebraic form with exponent γ and is $(z-1)/z$ if the RPD is fixed at the tangent point. The results are confirmed by numerical simulations.

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I. INTRODUCTION

Intermittency, which is a continuous route to chaos among a few known [1–3], is a signal that alternates randomly between a long regular (laminar) phase and relatively short irregular bursts. Pomeau and Manneville [4] classified the intermittency in one-dimensional maps into three types according to the way in which a fixed point loses its stability. Typical local Poincaré maps for type-I, -II and -III intermittencies are $y_{n+1}=y_n+uy_n^2+\epsilon$ $(u,\epsilon>0)$, $y_{n+1}=(1+\epsilon)y_n+uy_n^3$ $(\epsilon,u,y>0)$ [5], and $y_{n+1}=-(1+\epsilon)y_n-uy_n^3$ $(\epsilon,u>0)$ or $y_{n+2}=(1+2\epsilon)y_n+2u(1+2\epsilon)y_n^3$ $(\epsilon,u,y>0)$, respectively. Type-I intermittency has an algebraic type of characteristic relation between the average laminar length $\langle l\rangle$ and the channel width ϵ, $\langle l\rangle\propto\epsilon^{-1/2}$. For type-II and -III intermittencies, the origin is a fixed point and therefore the lower bound of reinjection (LBR) should be above the origin for the intermittency to appear. The characteristic relation of these types of intermittency is $\langle l\rangle\propto\ln(1/\epsilon)$ in the limit $\epsilon(\ll \text{LBR})\to 0$. We note that, among the three types of intermittency, type-I intermittency alone has a scaling property since it has an algebraic type of characteristic relation, thereby allowing the advent of renormalization group (RG) analysis.

RG analysis of a generalization of type-I intermittency was first performed by Hirsch, Nauenberg, and Scalapino [6]. They found that there exists a universal map $y_{n+1}=f^*(y_n)$ for intermittency, which is invariant under the Feigenbaum doubling operation [1] $Tf^*(y)=\alpha f^*(f^*(y/\alpha))=f^*(y)$ and satisfies the boundary condition $f^*(y\to 0)=y+u|y|^z$, where $z>1$. They further classified the perturbations to $f^*(y)$ according to their relevance. A function $f_\epsilon(y)=f^*(y)+\epsilon h_\lambda(y)h_\lambda(y)$ if $h_\lambda(y)$ is an eigenfunction with eigenvalue λ of the linearized doubling operator $L_f h_\lambda(y)=\alpha\{f^{*'}(f^*(y))h_\lambda(y)+h_\lambda(f^*(y))\}=\lambda h_\lambda(\alpha y)$. Hirsch *et al.* solved these equations using series expansion techniques, and Hu and Rudnick [7] found exact solutions using an implicit function technique. The results are as follows: $f^*(y)=[|y|^{-(z-1)}-d]^{-1/(z-1)}$; $h_\lambda(y)=[|y|^{-(z-1)}-u(z-1)]^{-z/(z-1)}\{|y|^{-p}-[|y|^{-(z-1)}-u(z-1)]^{p/(z-1)}\}/(pu)$; $\alpha=2^{1/(z-1)}$; $\lambda=2^{(p+1-z)/(z-1)}$, where $d=(z-1)u$ and $h_\lambda(|y|\to 0)=|y|^{2z-p-1}=|y|^q$.

With these results Hirsch *et al.* reached the characteristic relation $\langle l\rangle\propto\epsilon^{-\nu}$ with the critical exponent $\nu=(z-1)/z$ for the constant eigenperturbation [8]. Note that, for nonconstant eigenperturbations, the LBR should be above the origin as in type-II and -III intermittencies and therefore the characteristic relations are not of an algebraic form in the limit $\epsilon\to 0$. In other words, only the case of constant eigenperturbation has a scaling property and can be analyzed via the RG method.

In both Pomeau and Manneville's seminal work and Hirsch *et al.*'s RG analysis, the authors did not consider seriously the effect of the reinjection probability distribution (RPD) on the scaling properties of intermittency. It was, however, recently shown that [9], via numerical study of an example of type-I intermittency, if the LBR is at the tangent point, there appear various types of characteristic relation such that $\langle l\rangle\propto\epsilon^{-\nu}$ $(0\le\nu\le 1/2)$ when the RPD is a decreasing function of y and $\langle l\rangle\propto\ln(1/\epsilon)$ when the RPD is uniform. Such an algebraic form of characteristic relation implies that, even in the case of nonuniform RPD, some scaling properties exist and therefore an RG analysis is in order. In this Brief Report we derive the formula of the critical exponent for each case where the LBR is below or at the tangent point, by considering not only the universal map but also the effect of the RPD. We also examine numerically three maps of which each critical exponent is 1/2, 3/8, and 1/4 when the LBR is at the tangent point. The results of RG analysis agree well with the numerical ones.

II. CLASSIFICATION OF THE REINJECTION PROBABILITY DISTRIBUTION

We first classify the RPD according to the position of the LBR. We note that the laminar lengths for all reinjections below the lower bound of the gate are just the same as the laminar length for the reinjection at the lower bound of the gate itself (the gate set an acceptance $|y| \leq c$ on deviations in the laminar phase). The average laminar length $\langle l \rangle$ for a given RPD $P(y)$ is therefore given by

$$\langle l \rangle = l(-c, c) \int_{-\Delta}^{-c} P(y_{\text{in}}) dy_{\text{in}} + \int_{-c}^{c} l(y_{\text{in}}, c) P(y_{\text{in}}) dy_{\text{in}}, \tag{1}$$

where $-\Delta$ is the value of y_{in} representing the LBR and $l(y_{\text{in}}, c)$ is the laminar length for a reinjection at y_{in}. In case that the LBR is below the tangent point (the origin), the contribution of the second term on the right-hand side of Eq. (1) becomes negligible in the limit $\epsilon \ll c \to 0$ and therefore all RPD's have the same effective RPD,

$$P(y) = \delta(y + c). \tag{2}$$

We here normalize $P(y_{\text{in}})$, $\int_{-c}^{c} P(y_{\text{in}}) dy_{\text{in}} = 1$. When the LBR is at the tangent point, Eq. (1) simply reduces to $\langle l \rangle = \int_0^c l(y_{\text{in}}, c) P(y_{\text{in}}) dy_{\text{in}}$. Since we can assume without loss of generality that the RPD is a decreasing function of y [10] and the only leading order term of the RPD is relevant in the limit $\epsilon \ll c \to 0$, the RPD $P(y)$ can be written in general such that

$$P(y) = \frac{1}{y^\gamma} \qquad \text{or} \qquad \delta(y), \tag{3}$$

where $0 < \gamma < 1$ [$P(y)$ is non-normalizable if $\gamma \geq 1$]. The case where the LBR is above the tangent point is of little interest since the laminar length becomes zero as $\epsilon \ll c \to 0$.

III. DERIVATION OF THE CRITICAL EXPONENT OF INTERMITTENCY

We now derive the critical exponents of intermittency. A given difference equation $y_{n+1} = f(y_n)$ can be replaced by a differential equation

$$\frac{dy}{dl} = f(y) - y \tag{4}$$

in the long laminar length approximation. Here l is the number of iterations in the laminar phase. The average laminar length $\langle l \rangle [f]$ for the map $f(y)$ is

$$\langle l \rangle [f] = \int_{-c}^{c} dy_{\text{in}} P_f(y_{\text{in}}) \int_{y_{\text{in}}}^{c} \frac{dy}{f(y) - y}, \tag{5}$$

where $P_f(y)$ is the RPD for $f(y)$. Since $\langle l \rangle$ is related to the number of iterations and $f^2(y) = f(f(y))$ requires only half as many steps as $f(y)$, $\langle l \rangle [f^2] = (1/2) \langle l \rangle [f]$ if $P_{f^2}(y) = P_f(y)$, which is the case in general.

It can be shown that the normalized RPD's for $f(y)$, $f^2(y)$, and $Tf(y)$ are all the same by using the Shaw relation [11] which determines the RPD. The RPD is determined not by the local Poincaré map $f(y)$ but by the global map $F(A; x)$, from which the local Poincaré map comes. Here A is a tunable parameter and $y = -(x - x_c)/[\partial F(A, x_c)/\partial A]$ where x_c is the tangent point. The Shaw relation for the map $f(y)$ is $P_f(y)|dF/dy|_{y_i} = P_f(y_i)$ where $F(y_i) = y$. Here we use the Einstein summation convention. We may get $P_f(y)$ up to normalization by solving this equation. We note that $P_f(y)$ can also be expressed in terms of the probabilities at the points two iterations before, i.e., $P_f(y)|dF^2/dy|_{y_j} = P_f(y_j)$ where $F^2(y_j) = y$. Since this equation is the same as that of $P_{f^2}(y)$, the RPD's for $f(y)$ and $f^2(y)$ are equivalent up to normalization and become exactly the same after normalization. We next discuss $P_{Tf}(y)$. Note that $y_{n+1} = TF(y_n) = \alpha F^2(y_n/\alpha)$ can be rewritten as $(y_{n+1}/\alpha) = F^2(y_n/\alpha)$; that is, that the shape of $TF(y)$ in the y_n-y_{n+1} plane is the same as that of $F^2(y)$ in the αy_n-αy_{n+1} plane. Since the RPD is determined by the shape of the first return map we can immediately infer that $P_{Tf}(\alpha y) = \alpha P_{f^2}(y) = \alpha P_f(y)$ or $P_{Tf}(y) = \alpha P_f(y/\alpha)$, and therefore, for general forms of effective RPD's in Eqs. (2) and (3), the normalized $P_{Tf}(y)$ is equal to $P_f(y)$, i.e., $P_{Tf}(y)/\int_{-c}^{c} P_{Tf}(y)dy = P_f(y)/\int_{-c}^{c} P_f(y)dy$.

In order to derive critical exponents using the already given $f^*(y)$ and $h_\lambda(y)$, $\langle l \rangle [Tf]$ should be related to $\langle l \rangle [f]$. Since $P_{Tf}(y) = P_f(y)$, $\langle l \rangle [Tf]$ is given by

$$\begin{aligned} \langle l \rangle [Tf] &= \int_{-c}^{c} dy_{\text{in}} P_f(y_{\text{in}}) \int_{y_{\text{in}}}^{c} \frac{dy}{Tf(y) - y} \\ &= \alpha \int_{-c/\alpha}^{c/\alpha} dy_{\text{in}} P_f(\alpha y_{\text{in}}) \int_{y_{\text{in}}}^{c/\alpha} \frac{dy}{f^2(y) - y}. \end{aligned} \tag{6}$$

It is very likely that the average laminar length for a given map and RPD is independent of the scaling of the gate as long as ϵ is small enough. We present some examples to support this argument. In Fig. 1, relations between the average laminar length and the size of the gate are drawn in log-log scale, where the local Poincaré map is the typical one of type-I intermittency with $z = 2$ and $u = 1$. The RPD's for lines I and II and for lines III and IV are $\delta(y + c)$ and $1/\sqrt{y}$, respectively. The values of ϵ for lines I and III and for lines II and IV are 10^{-20} and 10^{-10}. These lines show that the smaller the value of ϵ, the longer the length of the flat region. That is, the average laminar length is maintained constantly though the gate size is reduced, in the limit $\epsilon \to 0$. If we accept the numerically supported argument, the final form of $\langle l \rangle [Tf]$ is

$$\langle l \rangle [Tf] = \alpha \int_{-c}^{c} dy_{\text{in}} P_f(\alpha y_{\text{in}}) \int_{y_{\text{in}}}^{c} \frac{dy}{f^2(y) - y}, \tag{7}$$

which is essential to calculate critical exponents of intermittency.

By using the relation in Eq. (7) we can obtain the formulas of critical exponents for the cases where the LBR is below and at the tangent point. We first consider the case where the LBR is below the tangent point. As mentioned above, in this case, the effective RPD is fixed at the lower bound of the gate in the limit $\epsilon \ll c \to 0$. Note that $l(-c,c) = 2l(0,c)$ due to the symmetry of the local Poincaré map and therefore

$$\int_{-c}^{c} dy_{\rm in}\delta(y+c)\int_{y_{\rm in}}^{c}\frac{dy}{f(y)-y}$$

$$= 2\int_{-c}^{c} dy_{\rm in}\delta(y_{\rm in})\int_{y_{\rm in}}^{c}\frac{dy}{f(y)-y}. \quad (8)$$

Since $\delta(\alpha y) = (\alpha)^{-1}\delta(y)$, $\langle l\rangle[Tf] = \alpha^{-1}\alpha\langle l\rangle[f^2] = 2^{-1}\langle l\rangle[f]$. After n iterations of this step we get $\langle l\rangle[f] = 2^n\langle l\rangle[T^n f] = 2^n\langle l\rangle[f^* + \lambda^n\epsilon h_\lambda]$. If we set $\lambda^n\epsilon = 1$, $\langle l\rangle[f^*+h_\lambda]$ is independent of ϵ and the ϵ dependence of $\langle l\rangle[f]$ is isolated to the prefactor 2^n. Since $\lambda = 2^{z/(z-1)}$ for a map $f(y) = f^*(y)+\epsilon$, the critical exponent $\nu[\text{LBR} < 0]$ defined by $\langle l\rangle[f] \propto \epsilon^{-\nu}$ is

$$\nu[\text{LBR} < 0] = \frac{z-1}{z}. \quad (9)$$

This is the result that Hirsch *et al.* obtained under the assumption of homogeneity.

We next consider the case where the LBR is at the tangent point. When $P(y) = 1/y^\gamma$ $(0 < \gamma < 1)$, we obtain

$$\langle l\rangle[f] = (2\alpha^{\gamma-1})^n\langle l\rangle[Tf] \quad (10)$$

from the relation in Eq. (7). Since $(2\alpha^{\gamma-1})^n = (2^n)^{(z-\gamma-2)/(z-1)} \propto \epsilon^{(z+\gamma-2)/z}$ for constant eigenperturbation if we set $\lambda^n\epsilon = 1$, the critical exponent $\nu[\text{LBR} = 0]$ is

$$\nu[\text{LBR} = 0] = \frac{z+\gamma-2}{z}. \quad (11)$$

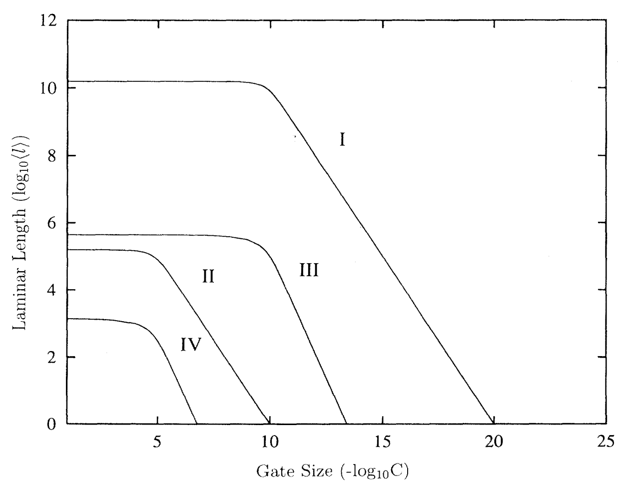

FIG. 1. Average laminar lengths vs the size of the gate for $y_{n+1} = y_n + y_n^2 + \epsilon$. The RPD's for lines I and II and for lines III and IV are $\delta(y+c)$ and $1/\sqrt{y}$, respectively. The values of ϵ for lines I and III and for lines II and IV are 10^{-20} and 10^{-10}, respectively. The smaller the value of ϵ, the longer the length of the flat region.

This is our main result. When the RPD is $P(y) = \delta(y)$, it has been shown that the critical exponent is given by Eq. (9).

IV. NUMERICAL RESULTS

In order to verify whether our reasonings used in the derivation of the critical exponents are correct, we compare, for three maps, the numerically obtained critical exponents with the ones obtained by the RG method. The first map is a quartic one, $x_{n+1} = f^{(A)}(f^{(B)}(x_n))$ where $f^{(A)}(x) = 4Ax(1-x)$ and $f^{(B)}(x) = 4Bx(1-x)$. The second map has a similar form to that of the first map except that $f^{(A)}(x) = A[1-16(x-1/2)]^4$ and

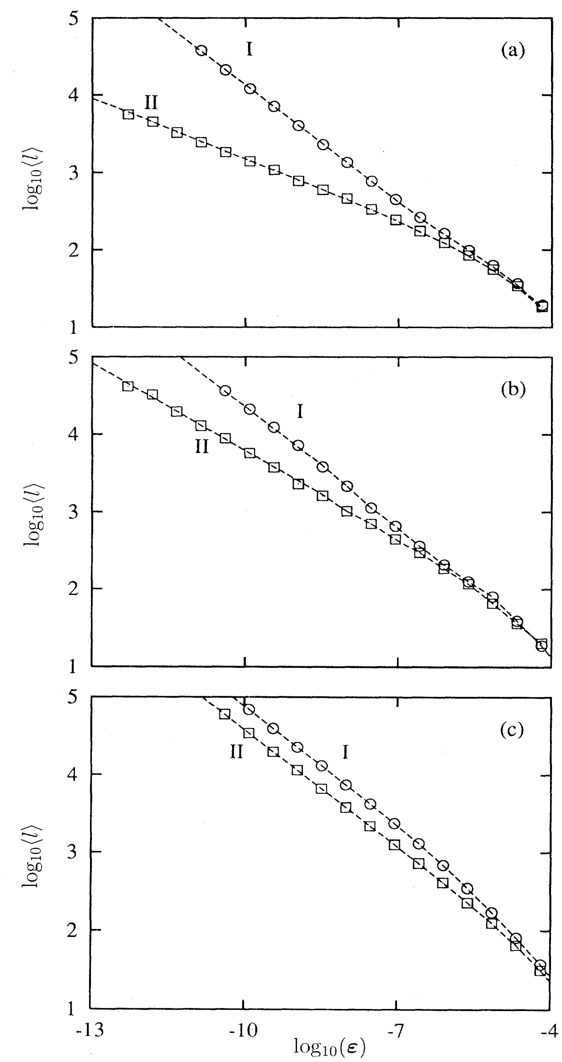

FIG. 2. Average laminar lengths vs ϵ. The local Poincaré map for these diagrams is the typical one of type-I intermittency. The RPD's for (a), (b), and (c) are $(y+\Delta)^{-1/2}$, $(y+\Delta)^{-3/4}$, and $\delta(y+\Delta)$, respectively. The critical exponents for these RPD's have the same value of 1/2 as long as the LBR is below the tangent point. When the LBR is at the tangent point, however, the critical exponent depends on the RPD: 1/4, 3/8, and 1/2 in (a), (b), and (c), respectively.

$f^{(B)}(x) = B[1 - 16(x - 1/2)]^4$. The third map has the same shape as that of the first map except in the range $a < x < b$, where a and b are some constants. In that range x_{n+1} has a constant value x^*. Local Poincaré maps for these maps, which are obtained by the Taylor expansion near each tangent point, have the typical form of type-I intermittency if they are expressed in terms of the coordinate y whose origin is the tangent point. Detailed procedures to obtain the local Poincaré map, the RPD, and the critical exponent, etc. are shown in our previous paper [9]. The RPD's obtained by solving the Shaw relation as well as by numerically counting reinjections are $(y+\Delta)^{-1/2}$, $(y+\Delta)^{-3/4}$, and $\delta(y+\Delta)$, respectively.

In Fig. 2(a) the characteristic relations for the first map are drawn. They are obtained in the range of small ϵ $(= B_c - B)$ for typical values of A. An intermittency appears last at the critical value B_c for a given value of A, at which the tangent bifurcation occurs. For the first map there are analytic relations among B_c, A, and the tangent point y_c: $B_c = [-9\sqrt{A} + 8A^{3/2} + (4A-3)^{3/2}]/16\sqrt{A}(A-1)$ and $y_c = 2/3 - (4 - 3/B_c)^{1/2}/6$. Line I is the characteristic relation whose critical exponent is $\nu = 1/2$ obtained for $A = 0.9415$, at which the LBR is below the tangent point. Line II showing $\nu = 1/4$ is obtained for $A = 0.941\,461\,95\ldots$, at which the LBR is at the tangent point. In Fig. 2(b) the characteristic relations for the second map are drawn. Line I representing $\nu = 1/2$ is obtained for $A = 0.981\,16$ and $B_c = 0.887\,009\,79\ldots$, at which the LBR is below the tangent point $y_c = 0.675\,202\,49\ldots$. Line II showing $\nu = 3/8$ is obtained for $A = 0.981\,153\,25\ldots$ and $B_c = 0.887\,008\,35\ldots$, at which the LBR is at the tangent point $y_c = 0.675\,203\,84\ldots$. In Fig. 2(c) the characteristic relations for the third map are drawn. For this map the tunable parameter to adjust the position of LBR is x^*. The values of A and B are fixed: $A = 0.941\,6$ and $B = 0.830\,230\,23\ldots$, at which the tangent point $y_c = 0.563\,045\,49\ldots$. We also fix the values a and b to be $a = 0.743$ and $b = 0.874$. Line I representing $\nu = 1/2$ is obtained for $x^* = 0.941\,6$, at which the LBR $(y = 0.562\,199\,7\ldots)$ is below the tangent point. Line II showing $\nu = 1/2$ is obtained for $x^* = 0.941\,479\,30\ldots$, at which the LBR is at the tangent point.

We note that, in all the above examples, the critical exponent is 1/2 as long as the LBR is below the tangent point although the RPD's are $(y+\Delta)^{-1/2}$, $(y+\Delta)^{-3/4}$, and $\delta(y+\Delta)$, respectively. This supports the theoretical result since $z = 2$ in these examples and therefore $\nu[\text{LBR} < 0] = 1/2$. We also note that, in the case that the LBR is at the tangent point, $\nu[\text{LBR} = 0] = 1/4$, 3/8, and 1/2 since $\gamma = 1/2$, 3/4, and $P(y) = \delta(y)$ for the first, second, and third maps.

V. CONCLUSION

We have shown explicitly that the critical exponent depends not only on the universal map but also on the RPD. When the LBR is below the tangent point all RPD's reduces to the one form of effective RPD, $P(y) = \delta(y + c)$. From this, the critical exponent is obtained such that $\nu[\text{LBR} < 0] = (z-1)/z$. On the other hand, when the LBR is at the tangent point the effective RPD is $1/y^\gamma$ $(0 < \gamma < 1)$ or $\delta(y)$. In this case, $\nu[\text{LBR} = 0] = (z + \gamma - 2)/z$ or $(z-1)/z$.

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[1] M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978).
[2] S. Newhouse, D. Ruelle, and F. Takens, Commun. Math. Phys. **64**, 35 (1978).
[3] C. Grebogi, E. Ott, and J. A. Yorke, Physica D **7**, 181 (1983).
[4] P. Manneville, and Y. Pomeau, Phys. Lett. **75A**, 1 (1979); Y. Pomeau, and P. Manneville, Commun. Math. Phys. **74**, 189 (1980).
[5] In Pomeau and Manneville's classification of intermittency, the stability matrix of type-II intermittency has complex engenvalues crossing the unit circle as ϵ changes. The typical Poincaré map composes of $\theta_{n+1} = \theta_n + \Omega$ as well as $y_{n+1} = (1+\epsilon)y_n + uy_n^3$ $(\epsilon, u, y > 0)$. However the radius variation alone can determine the average laminar length and consequently the characteristic relation.
[6] J. E. Hirsch, M. Nauenberg, and D. J. Scalapino, Phys. Lett. **87A**, 391 (1982).
[7] B. Hu and J. Rudnick, Phys. Rev. Lett. **48**, 1645 (1982).
[8] H. G. Schuster, *Deterministic Chaos*, 2nd revised ed. (VCH Publishers, Weinheim, 1987); Hao Bai-Lin, *Elementary Symbolic Dynamics and Chaos in Dissipative Systems* (World Scientific Publishing Co., Singapore, 1989).
[9] Chil-Min Kim, O. J. Kwon, Eok-Kyun Lee, and Hoyun Lee, Phys. Rev. Lett. **73**, 525 (1994).
[10] When the LBR is at the tangent point and the RPD is uniform the characteristic relation is logarithmic, which implies that there is no scaling property. This case is therefore of no interest from the viewpoint of RG analysis.
[11] R. Shaw, Z. Naturforsch Teil A **36**, 80 (1981); A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).